

# Instantons

A remarkable feature of the Yang-Mills action is that there are finite-action topological soliton solutions to the classical field equations. These solitons are known as instantons or, in early papers, as pseudoparticles. The Yang-Mills action is important in particle physics; in particular, it describes the behaviour of gluons, the particles which carry the strong nuclear force. Before instanton solutions were discovered in 1975 by Belavin, Polyakov, Schwarz and Tyupkin [Belavin *et al.* 1975], the Yang-Mills theory of the strong force appeared to have a symmetry not found in nature. This was known as the axial U(1) problem and was solved by 't Hooft who realised that one effect of the instanton solutions was to breaking this unwanted symmetry. This was the first example of an extended classical solution having a physical consequence in a field theory of particle physics.

In four euclidean dimensions the pure SU(2) Yang-Mills action is

$$S[A] = -\frac{1}{2} \int d^4x \text{trace } F_{\mu\nu} F_{\mu\nu} \quad (1)$$

where  $\mu$  and  $\nu$  are space indices running from 1 to 4 and repeated indices are summed.  $F_{\mu\nu}$  is a field strength tensor, a skewhermetian  $2 \times 2$  matrix field in four-dimensional space, related to a gauge potential  $A_\mu$  by

$$F_{\mu\nu} = \frac{\partial}{\partial x_\mu} A_\nu - \frac{\partial}{\partial x_\nu} A_\mu + [A_\mu, A_\nu] \quad (2)$$

The action is invariant under local gauge transformations

$$A_\mu \rightarrow A'_\mu = g A_\mu g^{-1} - \frac{\partial g}{\partial x_\mu} g^{-1} \quad (3)$$

where  $g$  is a special unitary matrix function in four dimensions. Under this transformation  $F_{\mu\nu} \rightarrow F'_{\mu\nu} = g F_{\mu\nu} g^{-1}$ . One key consequence of this is that  $F_{\mu\nu}$  not only vanishes when  $A_\mu = 0$ , but also, whenever  $A_\mu$  is a gauge transformation of zero. This means that the asymptotic condition required for finiteness of the action is that  $A_\mu$  approaches a gauge transformation of zero:

$$A_\mu \rightarrow -\frac{\partial g}{\partial x_\mu} g^{-1} \quad (4)$$

for  $r = \sqrt{x_\mu x_\mu} \rightarrow \infty$ . Now, the group of special unitary  $2 \times 2$  matrices is topologically a three-sphere and so the gauge potential on the large three-sphere at infinity gives a map from that three-sphere to the three-sphere of values of  $g$ . Maps between three-sphere are classified topologically by a single integer: the winding number. This integer, therefore, classifies finite-action field configurations; in fact, it is the only gauge-invariant information determined by the asymptotic field behaviour. The topologically nontrivial stationary points of the action  $S[A]$  are instantons.

Since they are stationary points of the action, instantons obey the Euler-Lagrange equation for the action:

$$\frac{\partial}{\partial x_\mu} F_{\mu\nu} = 0. \quad (5)$$

This is known as the Yang-Mills equation. It can be shown that minimal action solutions obey a first order equation called the self-dual equation:

$$F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\lambda\sigma} F_{\lambda\sigma} \quad (6)$$

where  $\epsilon_{\mu\nu\lambda\sigma}$  is totally skewsymmetric in its indices with  $\epsilon_{1234} = 1$ . Solutions to this equation obey the second order Yang-Mills equation. A solution with winding number one is given by

$$A_\mu = -\frac{r^2}{r^2 + R^2} \frac{\partial g}{\partial x_\mu} g^{-1} \quad (7)$$

where  $R$  is an arbitrary scale and  $g$  has the hedgehog form

$$g = \frac{x_4}{r} \mathbf{I}_2 + \frac{i}{r} (x_1 \sigma_1 + x_2 \sigma_2 + x_3 \sigma_3) \quad (8)$$

where  $\mathbf{I}_2$  is the  $2 \times 2$  identity matrix and the  $\sigma_i$ s are the Pauli matrices. Allowing for gauge equivalence, there is an eight-dimensional space of winding number one instantons, four of these dimensions correspond to the choice of position in space, one to the choice of scale and three to an overall group orientation. These results are particular to  $SU(2)$ , other groups can be studied; the main difference is in the number of overall group orientation parameters.

The self-dual equations are integrable. In fact, many of the integrable equations of mathematical physics can be derived from the self-dual equations by demanding that the solution has some translational or rotational

symmetry. Because of this integrability, solutions to the self-dual equations with winding numbers greater than one may be constructed, in principal, by twistor theory and by algebraic methods [Christ *et al.* 1978]. There is also a simple ansatz [Jackiw *et al.* 1976] but this does not give a general solution.

In the 1980s the self-dual equations revolutionised the study of smooth four-dimensional manifolds: by studying the space of solutions to the self-dual equations over different manifolds, it is possible to derive invariants, known as the Donaldson invariants [Donaldson & Kronheimer 1990]. This approach has since been largely superseded by Seiberg-Witten theory.

In the path integral approach to quantum field theory, physical values are derived by certain weighted integrations over all possible field configurations. These path integrals are often intractable and it is common to expand the integration about the stationary points of the action. This works because of the way the action appears in the integrand of the path integral. Calculations of this type are known as semi-classical calculations because they are effectively an expansion in Planck's constant,  $\hbar$ . This expansion must include a sum over all possible stationary points and so, in Yang-Mills theory, it includes a sum over instanton configurations. By studying the symmetry properties of the measure in Quantum Chromodynamics, the Yang-Mills theory describing the strong nuclear force, it can be shown that terms in this expansion break the axial U(1) symmetry. This symmetry is unbroken if the instanton terms are omitted. The symmetry breaking allows processes that violate baryon and lepton conservation, however, the amplitudes for these effects are highly suppressed. One useful approach is to consider these processes as tunnelling events between different vacua, in fact, instanton calculations in quantum field theory are very similar to WKB calculations in quantum mechanics.

While instantons provide a qualitative explanation for a host of phenomena in Quantum Chromodynamics, useful quantitative results are not available within the semi-classical approach and, even qualitatively, instanton calculations are not rigorous since they relate only to the semi-classical approximation; a truncation of the full quantum theory. The modern approach to Quantum Chromodynamics is lattice QCD. It is possible within lattice QCD to verify the original ideas about the role of instantons in the physics of the strong force, however, there are limits to the precision with which the lattice and semi-classical approaches can be compared [Negele 1998].

Finite action soliton solutions in other equation systems are sometimes referred to as instantons. Examples include the finite action solutions to

the forced Burger equation which arises in the study of turbulence and the vortex-like solutions in the Abelian Higgs model which is related to condensed matter physics. Instantons have many similarities to the lump solitons found in certain sigma models.

Conor Houghton

Cross reference to entries on Solitons, Yang-Mills theory, Quantum Field theory, Integrable models.

## References

- [Belavin *et al.* 1975] Belavin, A.A., Polyakov, A.M., Schwarz A.S & Tyupkin, Yu.S. 1975, Pseudoparticle solutions of the Yang-Mills equations, *Physics Letters B*, 59: 85-87. The original reference for the instanton solutions.
- [Christ *et al.* 1978] Christ, N.H., Weinberg, E.J. & Stanton, N.K. 1978, General self-dual Yang-Mills solutions, *Physical Review D* 18: 2013-2025. Includes a review of the complete solution to the self-dual equations originally due to Atiyah, Hitchin, Drinfeld and Manin.
- [Coleman 1985] Coleman, S. 1985, *Aspects of symmetries*. Cambridge: Cambridge University Press. Chapter 7 gives a celebrated description of instantons in QCD.
- [Donaldson & Kronheimer 1990] Donaldson, S.K. & Kronheimer, P.B. 1990 *The geometry of four-manifolds* Oxford: Oxford University Press. Classic book about Donaldson theory.
- [Jackiw *et al.* 1976] Jackiw R., Nohl C. & Rebbi C. 1977. Conformal properties of pseudoparticle configurations. *Physical Review D*, 15: 1642-1646. A large, but incomplete, family of solutions to the self-dual equations.

- [Negele 1998] Negele, J.W. 1998, Instantons, the QCD vacuum, and hadronic physics, *Nuclear Physics B Proceedings Supplements* , 73: 92-104. A review of instanton effects in lattice QCD.
- [’t Hooft 1976] ’t Hooft, G. 1976. Symmetry breaking through Bell-Jackiw anomalies, *Physical Review Letters*, 37: 8-11. The original reference for the solution of the axial  $U(1)$  problem.
- [Weinberg 1996] Weinberg, S. 1996, Quantum field theory, volume II. Cambridge: Cambridge University Press. Chapter 23 concerns extended field configurations in particle physics and has a very clear treatment of instantons.