Yang-Mills theory

Modern particle theories, such as the Standard model, are quantum Yang-Mills theories. In a quantum field theory, space-time fields with relativistic field equations are quantized and, in many calculations, the quanta of the fields are interpreted as particles. In a Yang-Mills theory these fields have an internal symmetry: they are acted on by a space-time dependant non-Abelian group transformations in a way that leaves physical quantities, such as the action, invariant. These transformations are known as local gauge transformations and Yang-Mills theories are also known as non-Abelian gauge theories.

Yang-Mills theories, and especially quantum Yang-Mills theories, have many subtle and surprising properties and are still not fully understood, either in terms of their mathematically foundation or in terms of their physical predictions. However, the importance of Yang-Mills theory is clear, the Standard Model has produced calculations of amazing accuracy in particle physics and, in mathematics, ideas arising from Yang-Mills theory and from quantum field theory, are increasingly important in geometry, algebra and analysis.

Consider a complex doublet scalar field ϕ_a; a scalar field is one that has no Lorentz index, but, as a doublet, ϕ_a transforms under a representation of SU(2), the group represented by special unitary 2×2 matrices:

$$\phi_a(x) \rightarrow g_{ab}\phi_b x \tag{1}$$

where $g \in$ SU(2) and the repeated index is summed over. If this is a global transformation, that is, if g is independent of x, then derivative of ϕ_a have the same transformation property as ϕ_a itself:

$$\frac{\partial \phi_a}{\partial x_\mu} \rightarrow \frac{\partial g_{ab}\phi_b}{\partial x_\mu} = g_{ab}\frac{\partial \phi_b}{\partial x_\mu} \tag{2}$$

However, this is not true for a local, or space-time dependant, transformations where

$$\frac{\partial \phi_a}{\partial x_\mu} \rightarrow \frac{\partial g_{ab}\phi_b}{\partial x_\mu} = g_{ab}\frac{\partial \phi_b}{\partial x_\mu} + \frac{\partial g_{ab}}{\partial x_\mu}\phi_b \tag{3}$$

In order to construct an action which includes derivatives and which is invariant under local transformations, a new derivative is defined which transforms

the same way as ϕ_a:

$$D_\mu \phi_a = \frac{\partial \phi_a}{\partial x_\mu} + (A_\mu)_{ab} \phi_b \tag{4}$$

where A_μ is a new two-indexed space-time field, called a gauge field or gauge potential, defined to have the transformation property

$$(A_\mu)_{ab} \rightarrow g_{ac}(A_\mu)_{cd} g_{db}^{-1} - \frac{\partial g_{ac}}{\partial x_\mu} g_{cb}^{-1} \tag{5}$$

Now, under a local transformation

$$D_\mu \phi_a \rightarrow g_{ab} D_\mu \phi_b \tag{6}$$

and so, $D_\mu \phi_a$ transforms in the same way as ϕ_a. This derivative is called a covariant derivative.

A physical theory which includes the gauge field A_μ should treat A_μ as a dynamical field and so the action should have a kinetic term for A_μ. In other words, the action should include derivative terms for A_μ. These terms are found in the field strength

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} + [A_\mu, A_\nu] \tag{7}$$

which has the covariant transformation property

$$(F_{\mu\nu})_{ab} \rightarrow g_{ac}(F_{\mu\nu})_{cd} g_{db}^{-1} \tag{8}$$

where $[A_\mu, A_\mu]$ is the normal matrix commutator. In fact, the simplest Yang-Mills theory is pure Yang-Mills theory with action

$$S[A] = -\frac{1}{2} \int d^4x \, \text{trace} \, F_{\mu\nu} F^{\mu\nu}. \tag{9}$$

and corresponding field equation

$$\frac{\partial F_{\mu\nu}}{\partial x_\mu} = 0 \tag{10}$$

Solutions to this equation are known as instantons.

More generally, Yang-Mills theories contain gauge fields and matter fields like ϕ and fields with both group and Lorentz or spinor indices. Also, the

group action described here can be generalized to other groups and to other representations. In the case of the Standard Model of particle physics, the gauge group is SU(3)×SU(2)×U(1) and the group representation structure is quite intricate.

Yang-Mills theory was first discovered in the 1950s, at this time, Quantum Electrodynamics was known to describe electromagnetism. Quantum Electrodynamics is a local gauge theory, but with an abelian gauge group. It was also known that there is a approximate global non-Abelian symmetry called isospin symmetry which acts on the proton and neutron fields as a doublet and on the pion fields as a triplet. This suggested that a local version of the isospin symmetry might give a quantum field theory for the strong force with the pions fields as gauge fields [O'Raifeartaigh, 1997]. This did not work because pion fields are massive whereas gauge fields are massless and the main thrust of theoretical effort in the 1950s and 1960s was directed at other models of particle physics.

However, it is now known that the proton, neutron and pion are not fundamental particles, but are composed of quarks and that there is, in fact, a quantum Yang-Mills theory of the strong force with quark fields and gauge particles called gluons. Furthermore, it is now known that it is possible to introduce a particle, called a Higgs boson, to break the non-Abelian gauge symmetry in the physics of a symmetric action and give mass terms for gauge fields. This mechanism is part of the Weinberg-Salam model, a quantum Yang-Mills theory of the electroweak force which is a component of the Standard Model and which includes both massive and massless gauge particles.

These theories were only discovered after several key experimental and theoretical breakthroughs in the late 1960s and early 1970s. After it became clear from collider experiments that proton have a substructure, theoretical study of the distance dependant properties of quantum Yang-Mills theory lead to the discovery that Yang-Mills fields are asymptotically free [Gross, 1999]. This means that the high-energy behaviour of Yang-Mills fields includes the particle like properties seen in experiments, but the low-energy behaviour may be quite different and, in fact, the quantum behaviour might not be easily deduced from the classical action. Confinement and the mass gap are examples of this. The strong force is a local gauge theory with quark fields. The quark structure of particles is observed in collider experiments, but free quarks are never detected, instead, at low-energies, they appear to

bind together to form composite particles, such as neutrons, protons and pions. This is called confinement. It is possible to observe this behaviour in simulations of the quantum gauge theory of the strong force, but it has not been possible to prove mathematically that confinement is a consequence of the theory. The same is true of the mass gap, it is known that particles have non-zero mass, and this is observed in simulations, but, there is no known way of deriving the mass gap mathematically from the original theory [Clay, 2002].

The symmetries of Yang-Mills theory can be extended to include a global symmetry between the bosonic and fermionic fields called supersymmetry. While there is no direct evidence for supersymmetry in physics, the indirect case is very persuasive and it is commonly believed that direct evidence will be found in the future. Often, supersymmetric theories are more tractable, for example, Seiberg and Witten have found exact formula for many quantum properties in $N = 2$ super-Yang-Mills theory [Seiberg & Witten 1994]. It is also commonly believed by theoretical physicists that the quantum Yang-Mills theories in particle physics are in fact a limit of a more fundamental string theory.

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Cross reference to entries on Instantons; Quantum Field theory; Matter, theory of; Higgs boson.

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